The Positive Semidefiniteness of Partitioned Matrices

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ABSTRACT

The positive semidefiniteness of a partitioned matrix is characterized in terms of its submatrices. The result is applied to a variety of problems concerning Löwner ordered matrices, which need not be partitioned themselves.

1. INTRODUCTION

Positive semidefinite (psd) matrices—also referred to as nonnegative definite matrices, Gramian matrices, covariance matrices, and dispersion matrices—play a prominent role in statistics. For example, the efficiency of estimators is formulated in terms of the difference of dispersion matrices. Sometimes prior knowledge about the distribution of measurement errors takes the form of bounds on the covariance matrix of the errors. Even entire stochastic models may be formulated by means of a structured covariance matrix of observable variables. In all these cases problems arise that deal with psd matrices, and often these problems can be solved by using the algebraic properties of psd matrices. Not seldom these problems relate to partitioned matrices directly. At other times, as will be shown, a reformulation in terms of partitioned matrices may prove itself useful. In any case the algebraic aspects of partitioned psd matrices are interesting in themselves.

In the next section the positive semidefiniteness of a partitioned matrix is formulated in terms of conditions on its submatrices; this result, which was presented by Albert (1969, 1972), is closely related to general results on Schur complements (cf. Ouellette, 1978). The usefulness of this basic result will be demonstrated throughout the paper. In the third section it will be

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shown that a Löwner ordering of matrices corresponds to the positive semidefiniteness of a partitioned matrix. Therefore the results on partitioned psd matrices can be applied to a variety of quite general problems not necessarily pertaining to partitioned matrices. Section 4 derives results on Löwner ordered partitioned matrices. In particular the theorems of Section 4 imply two propositions on bounds on regression coefficients in so-called errors-in-variables models, which were presented—without proof—by Bekker et al. (1987).

The paper uses the following convention. Matrices are real with a number of rows and columns that is in accordance with the expressions in which they appear. This holds in particular for expressions with partitioned matrices. That is, if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ c_{21} & C_{22} \end{bmatrix},$$

then, for example, $A_{11} = B_{11}C_{11} + B_{12}C_{12}$ is assumed to be a meaningful expression. Furthermore, if A is psd, i.e. $x'Ax \geqslant 0$ for all vectors x of an appropriate order, and A is symmetric, i.e. $A = A'$, where A' is the transpose of A, then A_{11} is assumed to be square, and thus psd. In that case we write $A \geqslant 0$, so that also $B'AB \geqslant 0$, and $B'AB = 0$ is equivalent to $AB = 0$. If $A \geqslant 0$ and A is nonsingular, then A is positive definite: $A > 0$. The Löwner partial ordering $A \geqslant B$ is a relation between symmetric matrices, meaning $A - B \geqslant 0$.

$\mathscr{R}(A)$ denotes the column space of A: the set of vectors Ax, where the vectors x are of an appropriate order. A^- denotes an arbitrary g-inverse of A, i.e. any matrix A^- such that $AA^-A = A$. So $B'A^-C$ is invariant under the choice of g-inverse if $\mathscr{R}(B) \subset \mathscr{R}(A')$ and $\mathscr{R}(C) \subset \mathscr{R}(A)$; cf. Rao and Mitra (1971, Lemma 2.2.4). Note that $\mathscr{R}(C) \subset \mathscr{R}(A)$ is equivalent to $AA^-C = C$, where A^- may be any g-inverse of A. A particular g-inverse is the Moore-Penrose inverse A^+, satisfying $AA^+A = A$, $A^+AA^+ = A^+$, $(A^+A)' = A^+A$, and $(AA^+)' = AA^+$. The Moore-Penrose inverse is unique, and for symmetric A it also satisfies $A^+A = AA^+$. The identity matrix is denoted by I.

2. A SINGLE PARTITIONED PSD MATRIX

A basic theorem for partitioned psd matrices has been stated by Albert (1969; 1972, Theorem 9.1.6).

Theorem 1 (Albert). *Let A be symmetric. Then:*

(a) $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \geqslant 0$ *if and only if*

(i) $A_{22} \geqslant 0$,
(ii) $A_{21} = A_{22}A_{22}^{-}A_{21}$,
(iii) $A_{11} \geqslant A_{12}A_{22}^{-}A_{21}$.

(b) $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} > 0$ *if and only if*

(i) $A_{22} > 0$,
(ii) $A_{11} > A_{12}A_{22}^{-1}A_{21}$.

Actually, Albert stated part (a) of the theorem using the Moore-Penrose inverse, but the generalization to g-inverses is not essential, since the relevant terms are invariant with respect to the choice of the g-inverse. Note that (a)(i), (ii), and (iii) could also have been formulated as: $A_{11} \geqslant 0$, $A_{12} = A_{11}A_{11}^{-}A_{12}$, $A_{22} \geqslant A_{21}A_{11}^{-}A_{12}$. In part (b) Albert added to the conditions (iii) $A_{11} > 0$, and (iv) $A_{22} > A_{21}A_{11}^{-1}A_{12}$; however, these are equivalent to the conditions (i) and (ii).

Consequently, every psd matrix E can be parametrized as

$$E = \begin{bmatrix} B'AB + C & B'A \\ AB & A \end{bmatrix}, \tag{1}$$

where $A \geqslant 0$, and $C \geqslant 0$. Furthermore if $A \geqslant 0$, and $C \geqslant 0$, then $E \geqslant 0$. Such a parametrization corresponds to a regression model. Let x and e be uncorrelated stochastic vectors with covariance matrices $\operatorname{cov}(x) = A$ and $\operatorname{cov}(e) = C$, respectively; then

$$y = B'x + e \tag{2}$$

implies that the covariance matrix of the stacked vector $(y', x')'$ is given by $E = \operatorname{cov}(y', x')'$.

The following lemma and theorem characterize the set of matrices B satisfying

$$\begin{bmatrix} C & B' \\ B & A \end{bmatrix} \geqslant 0.$$

LEMMA 1. *Let X be square, and $XX' \leqslant A$. Then*

$$\operatorname{tr}(X) \leqslant \operatorname{tr}(A^{1/2}),$$

with equality if $X = A^{1/2}$.

Proof. Let $X = P\Phi Q'$ with $P'P = PP' = I$, $Q'Q = QQ' = I$, and Φ diagonal; let $A = K\Psi K$, with $K'K = KK' = I$, and Ψ diagonal; let the diagonal elements of Φ and Ψ be arranged in descending order of magnitude. Then $P\Phi^2 P \leqslant K\Psi K'$ and $\Phi^2 \leqslant \Psi$ (cf. Beckenbach and Bellman, 1965, p. 73). Thus, $\operatorname{tr}(X) = \operatorname{tr}(P\Phi Q') = \operatorname{tr}(\Phi Q'P) \leqslant \operatorname{tr}\{(\Phi^2)^{1/2}\} \leqslant \operatorname{tr}(A^{1/2})$. Furthermore, $X = A^{1/2}$ satisfies $XX' \leqslant A$. ∎

THEOREM 2. *Let $A \geqslant 0$ and $Y'A^+Y \leqslant C$. Then*

$$\operatorname{tr}(R'AA^+Y) \leqslant \operatorname{tr}(C^{1/2}R'ARC^{1/2})^{1/2},$$

with equality if $Y = ARC^{1/2}\{(C^{1/2}R'ARC^{1/2})^{1/2}\}^+C^{1/2}$.

Proof. According to Theorem 1, $A \geqslant 0$ and $Y'A^+Y \leqslant C$ is equivalent to

$$\begin{bmatrix} C & Y'A^+A \\ AA^+Y & A \end{bmatrix} \geqslant 0.$$

Pre- and postmultiplication by $((I,0)',(0,R)')$ and its transpose, respectively shows that

$$\begin{bmatrix} C & Y'A^+AR \\ R'AA^+Y & R'AR \end{bmatrix} \geqslant 0.$$

So, according to Theorem 1, $R'AA^+YC^+Y'A^+AR \leqslant R'AR$, and also

$$C^{1/2}R'AA^+YC^+Y'A^+ARC^{1/2} \leqslant C^{1/2}R'ARC^{1/2}.$$

Furthermore, $R'AA^+YC^{+\,1/2}C^{1/2} = R'AA^+YC^+C = R'AA^+Y$. Therefore, by

Lemma 1,

$$\operatorname{tr}(R'AA^+Y) = \operatorname{tr}(R'AA^+YC^{+\,1/2}C^{1/2})$$

$$= \operatorname{tr}(C^{1/2}R'AA^+YC^{+\,1/2}) \leqslant \operatorname{tr}(C^{1/2}R'ARC^{1/2})^{1/2}.$$

Choosing $Y = ARC^{1/2}\{(C^{1/2}R'ARC^{1/2})^{1/2}\}^+C^{1/2}$, we find $\operatorname{tr}(R'AA^+Y) = \operatorname{tr}(C^{1/2}R'ARC^{1/2})^{1/2}$, and, as

$$\left\{(C^{1/2}R'ARC^{1/2})^{1/2}\right\}^+ C^{1/2}R'ARC^{1/2}\left\{(C^{1/2}R'AC^{1/2})^{1/2}\right\}^+ \leqslant I,$$

this Y also satisfies $Y'A^+Y \leqslant C$. ■

Theorem 2 gives an alternative characterization of the set of matrices B that satisfy

$$\begin{bmatrix} C & B' \\ B & A \end{bmatrix} \geqslant 0. \tag{3}$$

This can be seen as follows. Theorem 2 says that for all possible R

$$\operatorname{tr}(R'B) \leqslant \operatorname{tr}(C^{1/2}R'ARC^{1/2})^{1/2}, \tag{4}$$

with equality holding for some B satisfying (3). Obviously, the set of matrices B satisfying (3) is convex, and so is the set of vectors $\operatorname{vec}(B)$, formed by a stacking of the column vectors of B. As $\operatorname{tr}(R'B) = \{\operatorname{vec}(R)\}'\operatorname{vec}(B)$, the inequalities in (4) describe the supporting hyperplanes of the convex set of vectors $\operatorname{vec}(B)$. These hyperplanes fully characterize the set of matrices B.

Corollary 2.1. *Let $A \geqslant 0$ and $C \geqslant 0$. Then*

$$\begin{bmatrix} C & B' \\ B & A \end{bmatrix} \geqslant 0 \tag{i}$$

if and only if for all possible R,

$$\operatorname{tr}(R'B) \leqslant \operatorname{tr}(C^{1/2}R'ARC^{1/2})^{1/2}.$$

COROLLARY 2.2. *Let $A \geqslant 0$ and $C \geqslant 0$. Then the set of matrices $F'BG$, where B satisfies*

$$\begin{bmatrix} C & B' \\ B & A \end{bmatrix} \geqslant 0, \tag{i}$$

is given by the set of matrices X satisfying

$$\begin{bmatrix} G'CG & X' \\ X & F'AF \end{bmatrix} \geqslant 0. \tag{ii}$$

Proof. The convex set of matrices $F'BG$ where B satisfies (i) is characterized by its supporting hyperplanes

$$\operatorname{tr}(R'F'BG) = \operatorname{tr}(GR'F'B) \leqslant \operatorname{tr}(C^{1/2}GR'F'AFRG'C^{1/2})^{1/2}.$$

The set of matrices X satisfying (ii) is characterized by

$$\operatorname{tr}(R'X) \leqslant \operatorname{tr}\left\{(G'CG)^{1/2}R'F'AFR(G'CG)^{1/2}\right\}^{1/2}.$$

In these two inequalities the right-hand sides are equal, and thus the result follows. ∎

Note that if

$$\begin{bmatrix} C & B' \\ B & A \end{bmatrix} \geqslant 0,$$

then the set of scalars $\alpha = f'Bg$, where f and g are vectors, is given by

$$-\operatorname{tr}(C^{1/2}gf'Afg'C^{1/2})^{1/2} \leqslant \alpha \leqslant \operatorname{tr}(C^{1/2}gf'Afg'C^{1/2})^{1/2}, \tag{5}$$

which can be written more simply as

$$-\{(g'Cg)(f'Af)\}^{1/2} \leqslant \alpha \leqslant \{(g'Cg)(f'Af)\}^{1/2}, \tag{6}$$

where (6) is in accordance with Corollary 2.2.

3. LOEWNER ORDERED MATRICES

THEOREM 3. *Let A and B be symmetric. Then $0 \leqslant B \leqslant A$ if and only if*

(i) $A \geqslant 0$,
(ii) $B = AA^- B$,
(iii) $B \geqslant BA^- B$.

Proof. Apply Theorem 1 to

$$\begin{bmatrix} B & B \\ B & A \end{bmatrix} \geqslant 0$$

and, equivalently,

$$\begin{bmatrix} A & B \\ B & B \end{bmatrix} \geqslant 0. \qquad \blacksquare$$

Again, the relevant terms are invariant with respect to the choice of g-inverse. Theorem 3 generalizes Theorem 3.5 by Gaffke and Krafft (1982), which says that $B \geqslant BA^+ B$ is implied by $0 \leqslant B \leqslant A$.

For $A \geqslant 0$, Baksalary et al. (1983, Corollary 3) show the equivalence of $A \geqslant B'(BA^+B')^+B$ and $\mathscr{R}(B'BA) \subset \mathscr{R}(A)$. This result also follows from Theorem 3, on recognizing that the former condition is equivalent to $B'(BA^+B')^+B = AA^+B'(BA^+B')^+B$, which means $\mathscr{R}(B'(BA^+B')^+B) \subset R(A)$, or equivalently, since $A \geqslant 0$, $\mathscr{R}(B'BA) \subset \mathscr{R}(A)$.

A result by Milliken and Akdeniz (1977) says that if $0 \leqslant B \leqslant A$, then a necessary and sufficient condition for $0 \leqslant A^+ \leqslant B^+$ to hold true is that $\operatorname{rank}(B) = \operatorname{rank}(A)$. The result is also given by Gaffke and Krafft (1982, Theorem 3.3). It follows from Theorem 3 that if both $0 \leqslant B < A$ and $0 \leqslant A^+ \leqslant B^+$ then $B = AA^+B$ and $A^+ = B^+BA^+$, so $\operatorname{rank}(A) = \operatorname{rank}(B)$. If, on the other hand, $0 \leqslant B \leqslant A$, then $B \geqslant BA^+B$ and $\mathscr{R}(B) \subset \mathscr{R}(A)$; if furthermore, $\operatorname{rank}(A) = \operatorname{rank}(B)$, then $\mathscr{R}(B) = \mathscr{R}(A)$, so $A^+ = B^+BA^+$; consequently $B \geqslant BA^+B$ implies $A^+ \leqslant B^+$, which establishes the result.

In order to evaluate the extremes of a linear function of a matrix B that satisfies $C \leqslant B \leqslant A$, we first consider the following theorem, which corresponds to Theorem 2.

THEOREM 4. *Let A and Z be symmetric, $A \geqslant 0$. If $ZA^+Z \leqslant A$, then*

$$\operatorname{tr}(R'AA^+Z) \leqslant \tfrac{1}{2}\operatorname{tr}\left\{A^{1/2}(R+R')A(R+R')A^{1/2}\right\}^{1/2},$$

with equality if

$$Z = A(R + R')A^{1/2}\left[\left\{A^{1/2}(R + R')A(R + R')A^{1/2}\right\}^{1/2}\right]^{+} A^{1/2}.$$

Proof. If Z is symmetric, $A \geqslant 0$, and $ZA^{+}Z \leqslant A$, then

$$\begin{bmatrix} A & ZA^{+}A \\ AA^{+}Z & A \end{bmatrix} \geqslant 0,$$

and so, according to Theorem 1, $AA^{+}Z = AA^{+}ZA^{+}A = ZA^{+}A$. Consequently, $\operatorname{tr}(R'AA^{+}Z) = \operatorname{tr}\{ZA^{+}AR') = \operatorname{tr}(RAA^{+}Z)$. It follows from Theorem 2 that

$$\begin{aligned} \operatorname{tr}(R'AA^{+}Z) &= \tfrac{1}{2}\operatorname{tr}\{(R + R')AA^{+}Z\} \\ &\leqslant \tfrac{1}{2}\operatorname{tr}\left\{A^{1/2}(R + R')A(R + R')A^{1/2}\right\}^{1/2} \end{aligned}$$

with equality if $Z = A(R + R')A^{1/2}[\{A^{1/2}(R + R')A(R + R')A^{1/2}\}^{1/2}]^{+}A^{1/2}$, which happens to be symmetric. ■

Theorem 4 characterizes the set of symmetric matrices B satisfying

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \geqslant 0.$$

Again this set of matrices B is convex and thus fully characterized by its supporting hyperplanes, $\operatorname{tr}(R'B) \leqslant \frac{1}{2}\{A^{1/2}(R + R')A(R + R')A^{1/2}\}^{1/2}$. Consequently, the following corollary has been proved.

Corollary 4. *Let* $A \geqslant 0$, *then*

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \geqslant 0, \quad \textit{and} \quad B \textit{ is symmetric} \tag{i}$$

if and only if for all possible R

$$\operatorname{tr}(R'B) \leqslant \tfrac{1}{2}\operatorname{tr}\left\{A^{1/2}(R + R')A(R + R')A^{1/2}\right\}^{1/2}. \tag{ii}$$

Indeed, the symmetry of B is implied by condition (ii). This can also be seen as follows. If B satisfies (ii) for all possible R, then it satisfies (ii) for all skew-symmetric R, which satisfy $R+R'=0$. So for all skew-symmetric R, $\operatorname{tr}(R'B)=0$, implying the symmetry of B.

The following corollary corresponds to Corollary 2.2; it can be proved in a similar way.

COROLLARY 4.2. *Let $A \geqslant 0$. Then the set of matrices $F'BF$ where B satisfies*

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \geqslant 0 \quad \textit{and} \quad B \textit{ is symmetric} \tag{i}$$

is given by the set of matrices X satisfying

$$\begin{bmatrix} F'AF & X \\ X & F'AF \end{bmatrix} \geqslant 0 \quad \textit{and} \quad X \textit{ is symmetric.} \tag{ii}$$

COROLLARY 4.3. *Let $A \geqslant 0$. Then the set of vectors Bf where B satisfies*

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \geqslant 0 \quad \textit{and} \quad B \textit{ is symmetric} \tag{i}$$

is given by the set of vectors x satisfying

$$\begin{bmatrix} f'Af & x' \\ x & A \end{bmatrix} \geqslant 0. \tag{ii}$$

Proof. The set of vectors Bf where B satisfies (i) is convex and thus fully characterized by its supporting hyperplanes

$$r'Bf = \operatorname{tr}(fr'B) \leqslant \tfrac{1}{2}\operatorname{tr}\left\{ A^{1/2}(fr'+rf')A(fr'+rf')A^{1/2}\right\}^{1/2}.$$

The convex set of vectors x satisfying (ii) is characterized by

$$r'x \leqslant (f'Afr'Ar)^{1/2}.$$

In these two inequalities the right-hand sides are equal—which can be verified by noting that the singular values of $\frac{1}{2}A^{1/2}(fr' + rf')A^{1/2}$ are given by $\frac{1}{2}\{r'Af \pm (f'Afr'Ar)^{1/2}\}$—and thus the result follows. ∎

The main result of the paper can now be formulated in the next theorem.

THEOREM 5. *Let A, B and C be symmetric, $C \leqslant A$, and let*

$$Q_R = \left\{(A-C)^{1/2}(R+R')(A-C)(R+R')(A-C)^{1/2}\right\}^{1/2}.$$

Then:

(5.1) *We have*

$$C \leqslant B \leqslant A, \tag{i}$$

if and only if for all possible R

$$\operatorname{tr}(R'B) \leqslant \tfrac{1}{2}\operatorname{tr}\{R'(A+C)\} + \tfrac{1}{4}\operatorname{tr}(Q_R). \tag{ii}$$

(5.2) *If*

$$B = \tfrac{1}{2}(A+C) + \tfrac{1}{2}(A-C)(R+R')(A-C)^{1/2}Q_R^{+}(A-C)^{1/2}, \tag{i}$$

then

$$C \leqslant B \leqslant A \quad \text{and} \quad \operatorname{tr}(R'B) = \tfrac{1}{2}\operatorname{tr}\{R'(A+C)\} + \tfrac{1}{4}\operatorname{tr} Q_R. \tag{ii}$$

(5.3) *The set of matrices $F'BF$, where B satisfies*

$$C \leqslant B \leqslant A, \tag{i}$$

is given by the set of matrices X satisfying

$$F'CF \leqslant X \leqslant F'AF. \tag{ii}$$

(5.4) *The set of vectors Bf, where f is a vector and B satisfies*

$$C \leqslant B \leqslant A, \tag{i}$$

is given by the set of vectors x satisfying

$$\begin{aligned}(A-C)(A-C)^{-}\left[x-\tfrac{1}{2}(A+C)f\right] &= x-\tfrac{1}{2}(A+C)f,\\ \left[x-\tfrac{1}{2}(A+C)f\right]'(A-C)^{-}\left[x-\tfrac{1}{2}(A+C)f\right] &\leqslant \tfrac{1}{4}f'(A-C)f.\end{aligned} \tag{ii}$$

Proof. Let

$$E \equiv \begin{bmatrix} A-C & 2B-(A+C) \\ 2B-(A+C) & A-C \end{bmatrix}.$$

Then $C \leqslant B \leqslant A$ is equivalent to $E \geqslant 0$, since the matrix $((I,I)',(I,-I)')$ is nonsingular, and

$$E = \begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} B-C & 0 \\ 0 & A-B \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}.$$

Consequently, (5.1) and (5.2) are direct implications of Corollary 4.1 and Theorem 4; (5.3) is implied by Corollary 4.2; and (5.4) can be verified using Corollary 4.3 and Theorem 1. ∎

N.B. As a consequence of Theorem 5(5.1) and (5.2), the set of scalars $\mathrm{tr}(R'B)$ where $C \leqslant B \leqslant A$ is given by the set of scalars α satisfying

$$\tfrac{1}{2}\mathrm{tr}\{R'(A+C)\} - \tfrac{1}{4}\mathrm{tr}(Q_R) \leqslant \alpha \leqslant \tfrac{1}{2}\mathrm{tr}\{R'(A+C)\} + \tfrac{1}{4}\mathrm{tr}(Q_R). \tag{7}$$

Let π be equal to the sum of positive eigenvalues of $\frac{1}{2}(A-C)^{1/2}(R+R')\times(A-C)^{1/2}$, and let ν be equal to the sum of its negative eigenvalues. Then the set of scalars α is also given by

$$\mathrm{tr}(R'C)+\nu \leqslant \alpha \leqslant \mathrm{tr}(R'C)+\pi. \tag{8}$$

COROLLARY 5.1. *Let A, B, and C be symmetric, $C \leqslant A$. Then the set of scalars $g'Bf$, where g and f are vectors and B satisfies*

$$C \leqslant B \leqslant A, \tag{i}$$

is given by the set of scalars α satisfying

$$\alpha \leqslant \tfrac{1}{2}(g'Af + g'Cf) + \tfrac{1}{2}\{g'(A-C)gf'(A-C)f\}^{1/2},$$

$$\alpha \geqslant \tfrac{1}{2}(g'Af + g'Cf) - \tfrac{1}{2}\{g'(A-C)gf'(A-C)f\}^{1/2}. \tag{ii}$$

Proof. The result follows from Theorem 5(5.4), Theorem 1, and Corollary 2.2. ■

Application. Theorem 5 can be used to derive bounds for regression coefficients when the independent variables are subject to measurement error with a bounded covariance matrix (cf. Bekker et al., 1984; Klepper and Leamer, 1984). This amounts to describing the set of vectors

$$\beta = (A - V)^{-1}Ab, \tag{9}$$

where V varies over $V_* \leqslant V \leqslant V^* < A$. Obviously, as follows from the abovementioned result by Milliken and Akdeniz, the matrix $(A - V)^{-1}$ is also bounded: $(A - V_*)^{-1} \leqslant (A - V)^{-1} \leqslant (A - V^*)^{-1}$. Therefore, if we let

$$H = (A - V^*)^{-1} - (A - V_*)^{-1}, \tag{10}$$

$$\beta_* = (A - V_*)^{-1}Ab, \qquad \beta^* = (A - V^*)^{-1}Ab, \tag{11}$$

the set of vectors β is given by the vectors in an ellipsoid

$$HH^-\left[\beta - \tfrac{1}{2}(\beta^* + \beta_*)\right] = \beta - \tfrac{1}{2}(\beta + \beta_*), \tag{12}$$

$$\left[\beta - \tfrac{1}{2}(\beta^* + \beta_*)\right]'H^-\left[\beta - \tfrac{1}{2}(\beta^* + \beta_*)\right] \leqslant \tfrac{1}{4}b'AHAb. \tag{13}$$

The extreme values of a linear combination $r'\beta$, where r is a vector, can be found by applying Corollary 5.1 to $r'(A - V)^{-1}Ab$. We find

$$\operatorname{extr} r'\beta = \tfrac{1}{2}r'(\beta^* + \beta_*) \pm \tfrac{1}{2}\{(b'AHAb)(r'Hr)\}^{1/2}. \tag{14}$$

4. LOEWNER ORDERED PARTITIONED MATRICES

In this section we consider positive definite matrices parametrized according to a regression model, or simply according to Theorem 1, i.e.

$$E=\begin{bmatrix} B'AB+C & B'A \\ AB & A \end{bmatrix} > 0. \tag{15}$$

Note that $E>0$ is equivalent to $A>0$ and $C>0$. According to the model (2), the elements of B correspond to regression coefficients. The following three theorems are motivated by the problem of describing the set of regression coefficients if the covariance matrix E is allowed to vary between bounds: $E_* \leqslant E \leqslant E^*$.

Let

$$E_*=\begin{bmatrix} B'_*A_*B_*+C_* & B'_*A_* \\ A_*B_* & A_* \end{bmatrix} \tag{16}$$

and

$$E^*=\begin{bmatrix} B^{*\prime}A^*B+C^* & B^{*\prime}A^* \\ A^*B^* & A^* \end{bmatrix} \tag{17}$$

LEMMA 2. *Let E, E_*, and E^* be positive definite matrices, parametrized as in* (15), (16), *and* (17), *respectively, then*

(2.1) *we have*

$$E \leqslant E^* \tag{i}$$

if and only if

$$\begin{bmatrix} C^*-C & B^{*\prime}-B' \\ B^*-B & A^{-1}-A^{*-1} \end{bmatrix} \geqslant 0. \tag{ii}$$

(2.2) *We have*

$$E_* \leqslant E \leqslant E^* \tag{i}$$

if and only if

$$0 \leqslant \begin{bmatrix} C - C_* & B' - B'_* \\ B - B_* & A_*^{-1} - A^{-1} \end{bmatrix} \leqslant \begin{bmatrix} C^* - C_* & B^{*\prime} - B'_* \\ B^* - B_* & A_*^{-1} - A^{*-1} \end{bmatrix}. \tag{ii}$$

Proof. For the first part of the lemma, premultiply E and E^* by $((I,0)',(-B,I)')$ and postmultiply by its transpose. Then the inequality $E \leqslant E^*$ is equivalent to

$$\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix} \leqslant \begin{bmatrix} (B^*-B)'A^*(B^*-B)+C^* & (B^*-B)'A^* \\ A^*(B^*-B) & A^* \end{bmatrix},$$

which is equivalent to an inequality between the inverses,

$$\begin{bmatrix} C^{*-1} & -C^{*-1}(B^*-B)' \\ -(B^*-B)C^{*-1} & A^{*-1}+(B^*-B)C^{*-1}(B^*-B)' \end{bmatrix} \leqslant \begin{bmatrix} C^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} -I \\ B^*-B \end{bmatrix} C^{*-1}[-I, B^{*\prime}-B'] \leqslant \begin{bmatrix} C^{-1} & 0 \\ 0 & A^{-1}-A^{*-1} \end{bmatrix}.$$

Applying Theorem 3, we find an equivalent formulation:

(i) $A^{-1} - A^{*-1} \geqslant 0$,
(ii) $(A^{-1} - A^{*-1})(A^{-1} - A^{*-1})^{-}(B^* - B) = (B^* - B)$,
(iii) $(B^* - B)'(A^{-1} - A^{*-1})^{-}(B^* - B) \leqslant C^* - C$,

which gives the result by application of Theorem 1. The second part of the lemma is a direct consequence of the first part. ∎

Thus, if E, E_*, and E^* are positive definite matrices, parametrized according to (15), (16), and (17), respectively, and $E_* \leqslant E \leqslant E^*$, then the set of matrices B is characterized by condition (ii) of Lemma 2(2.2). Theorem 5 can now be used, by applying it to this latter condition, to describe the set of matrices B. In order to present this last result, let

$$\tilde{R} = \begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix}, \tag{18}$$

where $\tilde{R}$ has been partitioned analogously to E in (15). Furthermore, let

$$K = \begin{bmatrix} C^* - C_* & B^{*\prime} - B'_* \\ B^* - B_* & A_*^{-1} - A^{*-1} \end{bmatrix}. \tag{19}$$

THEOREM 6. *Let E, E_*, and E^* be positive definite matrices, parametrized as in* (15), (16), *and* (17), *respectively*, $E_* \leqslant E^*$, *and let* $Q_{\tilde{R}} = \{K^{1/2}(\tilde{R} + \tilde{R}')K(\tilde{R} + \tilde{R}')K^{1/2}\}^{1/2}$, *where $\tilde{R}$ is parametrized as in* (18), *and K as in* (19). *Then*:

(6.1) *The set of scalars $tr(R'B)$, where E satisfies*

$$E_* \leqslant E \leqslant E^*, \tag{i}$$

is given by the set of scalars α satisfying

$$\begin{aligned} \alpha &\leqslant \tfrac{1}{2}\operatorname{tr}\{R'(B^* + B_*)\} + \tfrac{1}{4}\operatorname{tr}(Q_{\tilde{R}}), \\ \alpha &\geqslant \tfrac{1}{2}\operatorname{tr}\{R'(B^* + B_*)\} - \tfrac{1}{4}\operatorname{tr}(Q_{\tilde{R}}). \end{aligned} \tag{ii}$$

(6.2) *The set of vectors Bf, where f is a vector and E satisfies*

$$E_* \leqslant E \leqslant E^*, \tag{i}$$

is given by the set of vectors x satisfying

$$\begin{gathered} (A_*^{-1} - A^{*-1})(A_*^{-1} - A^{*-1})^{-}\left[x - \tfrac{1}{2}(B_* + B^*)f\right] = x - \tfrac{1}{2}(B_* + B^*)f, \\ \left[x - \tfrac{1}{2}(B_* + B^*)f\right]'(A_*^{-1} - A^{*-1})^{-}\left[x - \tfrac{1}{2}(B_* + B^*)f\right] \leqslant \tfrac{1}{4}f'(C^* - C_*)f. \end{gathered} \tag{ii}$$

(6.3) *The set of scalars $g'Bf$, where g and f are vectors and E satisfies*

$$E_* \leqslant E \leqslant E^*, \tag{i}$$

is given by the set of scalars α *satisfying*

$$\alpha \leqslant \tfrac{1}{2}g'(B_* + B^*)f + \tfrac{1}{2}\left\{g'(A_*^{-1} - A^{*-1})gf'(C^* - C_*)f\right\}^{1/2},$$

$$\alpha \leqslant \tfrac{1}{2}g'(B_* + B^*)f - \tfrac{1}{2}\left\{g'(A_*^{-1} - A^{*-1})gf'(C^* - C_*)f\right\}^{1/2}. \qquad \text{(ii)}$$

Proof. $E_* \leqslant E \leqslant E^*$ is equivalent to condition (ii) in Lemma 2(2.2).
(6.1): The set of scalars

$$\operatorname{tr}\left\{\begin{bmatrix} 0 & R' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C - C_* & B' - B'_* \\ B - B_* & A_*^{-1} - A^{-1} \end{bmatrix}\right\} = \operatorname{tr}\{R'(B - B_*)\}$$

is given, according to Theorem 5(5.1) and (5.2), by an expression analogous to (7). Hence the first result.

(6.2): According to Theorem 5(5.4) and Theorem 1, the set of vectors $(f'(C - C_*), f'(B' - B'_*))'$ is given by the set of vectors $(z', x' - f'B'_*)'$, satisfying

$$\begin{bmatrix} \tfrac{1}{4}f'(C^* - C_*)f & z' - \tfrac{1}{2}f'(C^* - C_*) & x' - \tfrac{1}{2}f'(B^* + B_*)' \\ z - \tfrac{1}{2}(C^* - C_*)f & C^* - C_* & B^{*\prime} - B'_* \\ x - \tfrac{1}{2}(B^* + B_*)f & B^* - B_* & A_*^{-1} - A^{*-1} \end{bmatrix} \geqslant 0.$$

According to Corollary 2.2, the set of vectors x is then given by the condition

$$\begin{bmatrix} \tfrac{1}{4}f'(C^* - C_*)f & x' - \tfrac{1}{2}f'(B^* + B_*)' \\ x - \tfrac{1}{2}(B^* + B_*)f & A_*^{-1} - A^{*-1} \end{bmatrix} \geqslant 0.$$

Thus by Theorem 1, the set of vectors Bf is given by the set of vectors x satisfying condition (ii) in (6.2).

(6.3): The last result is similar to Corollary 5.1. ■

Theorem 6 generalizes, and proves, Propositions 1 and 2 in Bekker et al. (1987). In that paper the regression model in (2) is considered, i.e.

$$y = B'x + e, \qquad (2)$$

where y and x are unobserved. Instead y^* and x^* are observed, and

$$\begin{aligned} y^* &= y + v, \\ x^* &= x + w, \end{aligned} \tag{20}$$

where v and w are measurement errors, which are assumed to be independent of y, x, and e.

Consequently, if M is the covariance matrix of (v', w'), $M = \operatorname{cov}(v', w')$, then $\operatorname{cov}(y', x') = \operatorname{cov}(y^{*\prime}, x^{*\prime}) - M$. The covariance matrix M is assumed to be bounded

$$0 \leqslant M \leqslant M^* < \operatorname{cov}(y^{*\prime}, x^{*\prime}); \tag{21}$$

hence

$$0 < \operatorname{cov}(y^{*\prime}, x^{*\prime}) - M^* \leqslant \operatorname{cov}(y', x') \leqslant \operatorname{cov}(y^{*\prime}, x^{*\prime}). \tag{22}$$

As was noted in Section 2, the covariance matrix $\operatorname{cov}(y', x') = E$ can be partitioned as in (1), and thus, given M^* and (a consistent estimate of) $\operatorname{cov}(y^{*\prime}, x^{*\prime})$, linear combinations of the elements of B can be (consistently) bounded by application of Theorem 6. For the special case that B consists of only one column, $B = b$, say, so that y is a scalar, Propositions 1 and 2 in Bekker et al. (1987) describe the set of vectors b, and the set of linear combinations $r'b$, respectively, analogously to Theorem 6.2 and 6.3.

Of course, Theorem 6 is much more general. If y is a vector of dependent variables, not necessarily consisting of only one element, then Theorem 6(6.1) gives the bounds on arbitrary linear combinations of the regression coefficients in the matrix B.

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